

ON THE KAUFFMAN BRACKET SKEIN MODULE OF SURGERY ON A $(2, 2b)$ TORUS LINK

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ABSTRACT. We show that the Kauffman bracket skein modules of certain manifolds obtained from integral surgery on a $(2, 2b)$ torus link are finitely generated, and list the generators for select examples.

1. INTRODUCTION

In [8], Kauffman presents an elegant construction of the Jones polynomial, an invariant of oriented links in S^3 , by constructing a new invariant, the Kauffman bracket polynomial. The Kauffman bracket is an invariant of unoriented framed links in S^3 , defined by the following skein relations:

$$\begin{aligned} (1) \quad \left\langle \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right\rangle &= A \left\langle \begin{array}{c} \diagup \\ \diagup \end{array} \right\rangle \left\langle \begin{array}{c} \diagdown \\ \diagdown \end{array} \right\rangle + A^{-1} \left\langle \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} \right\rangle \\ (2) \quad \langle L \cup \text{unknot} \rangle &= (-A^{-2} - A^2) \langle L \rangle \end{aligned}$$

For the invariant to be well-defined, one also must normalize it by choosing a value for the empty link. $\langle \text{empty link} \rangle = 1$, for instance.

Alternatively, we can use the skein relations to construct a module of equivalence classes of links in S^3 , or, for that matter, in any oriented 3-manifold. See Przytycki [10] and Turaev [13].

Definition 1. *Let N be an oriented 3-manifold, and let R be a commutative ring with identity, with a specified unit A . The Kauffman bracket skein module of N , denoted $S(N; R, A)$, or simply $S(N)$, is the free R -module generated by the framed isotopy classes of unoriented links in N , including the empty link, quotiented by the skein relations which define the Kauffman bracket.*

Since every crossing and unknot can be eliminated from a link in S^3 by the skein relations, $S(S^3)$ is generated by the empty link. Kauffman's argument that his bracket polynomial is well-defined shows that $S(S^3)$ is free on the empty link.

For $R = \mathbb{Z}[A^{\pm 1}]$, Hoste and Przytycki have computed the skein modules of all of the closed, oriented manifolds of genus 1: $S(L(p, q))$, which is free on $\lfloor \frac{p}{2} \rfloor + 1$ generators [6], and $S(S^1 \times S^2) \cong \mathbb{Z}[A^{\pm 1}] \oplus (\bigoplus_{i=1}^{\infty} \mathbb{Z}[A^{\pm 1}]/(1 - A^{2i+4}))$ [7]. Over $\mathbb{Z}[A^{\pm 1}]$, localized by inverting all of the cyclotomic polynomials, Gilmer and the author have computed the skein module of the quaternionic manifold [5].

Additionally, Bullock has determined whether or not the skein module obtained from integral surgery on a trefoil is finitely generated in [1]. In this paper, we pursue a similar result, for integral surgery on a $(2, 2b)$ torus link.

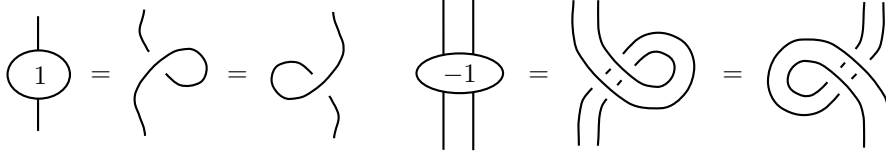
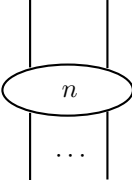
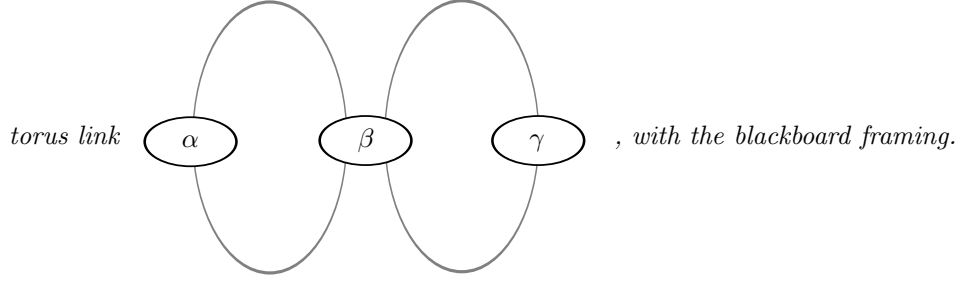


FIGURE 1. Examples of twist notation

Notation 2. For any integer n , let  denote n full twists in the depicted strands. For example, see Figure 1.

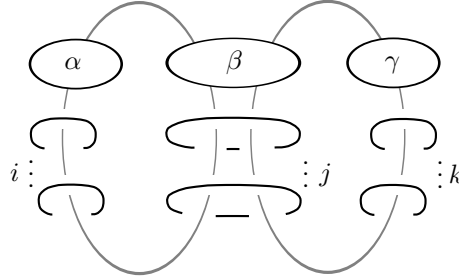
Definition 3. We define $M(\alpha, \beta, \gamma)$ to be the manifold obtained by surgery on the



Theorem 4. For all integers α , β , and γ such that $a = |\alpha| > 1$, $b = |\beta| > 1$, $c = |\gamma| > 1$, $\frac{1}{a} < \frac{1}{b} + \frac{1}{c}$, $\frac{1}{b} < \frac{1}{a} + \frac{1}{c}$, and $\frac{1}{c} < \frac{1}{a} + \frac{1}{b}$, $S(M(\alpha, \beta, \gamma))$ is finitely generated.

For specific values of α , β , and γ , we can use brute-force computation to refine our result, explicitly listing generating sets for $S(M(\alpha, \beta, \gamma))$.

Notation 5. We refer to the collection of loops



in $M(\alpha, \beta, \gamma)$ using the algebraic notation $x^i y^j z^k$.

In particular, we obtain the following for $S(M(2, -2, 2))$, $S(M(3, -2, 3))$, and $S(M(3, -2, 5))$ (the skein modules of the 3-fold, 4-fold, and 5-fold branched cyclic coverings of S^3 over the trefoil, respectively, as listed by Rolfsen [12]):

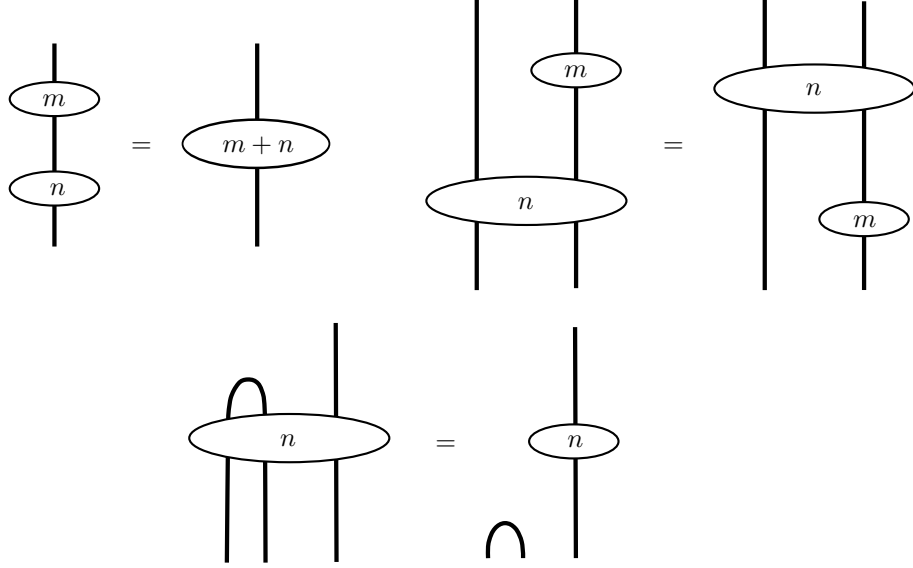


FIGURE 2. Useful properties of twists

| α | β | γ | <u>fundamental group</u> | <u>generators</u> |
|----------|---------|----------|--------------------------|---------------------------------------|
| 2 | -2 | 2 | quaternion group | $1, z, z^2, y, x$ |
| 3 | -2 | 3 | binary tetrahedral group | $1, z, z^2, z^3, y, x, x^2$ |
| 3 | -2 | 5 | binary icosahedral group | $1, z, z^2, z^3, z^4, z^5, y, x, x^2$ |

Note that the generating set for the skein module of the quaternionic manifold essentially coincides with what was shown in [5] over the ring R' obtained from $\mathbb{Z}[A^{\pm 1}]$ by inverting the multiplicative set generated by the elements of the set $\{A^n - 1 | n \in \mathbb{Z}^+\}$. Since any dependence relation over $\mathbb{Z}[A^{\pm 1}]$ would hold over R' and since $S(M(2, -2, 2); R', A)$ is a free module of rank 5, we obtain the following:

Corollary 6. $S(M(2, -2, 2); \mathbb{Z}[A^{\pm 1}], A)$ is a free module of rank 5.

2. TWISTS AND LOOPS

Twists have many useful properties, a few of which are listed in Figure 2. Note that, to obtain clearer diagrams, we represent a fixed but arbitrary number of parallel strands with a thick line.

We are most interested in using skein relations and isotopy to rewrite one strand, twisted with others, as a linear combination involving loops encircling the others, as in Figure 3.

In fact, repeating by repeating the steps performed in Figure 3, we obtain the following lemma:

Lemma 7. For each integer $n > 0$,

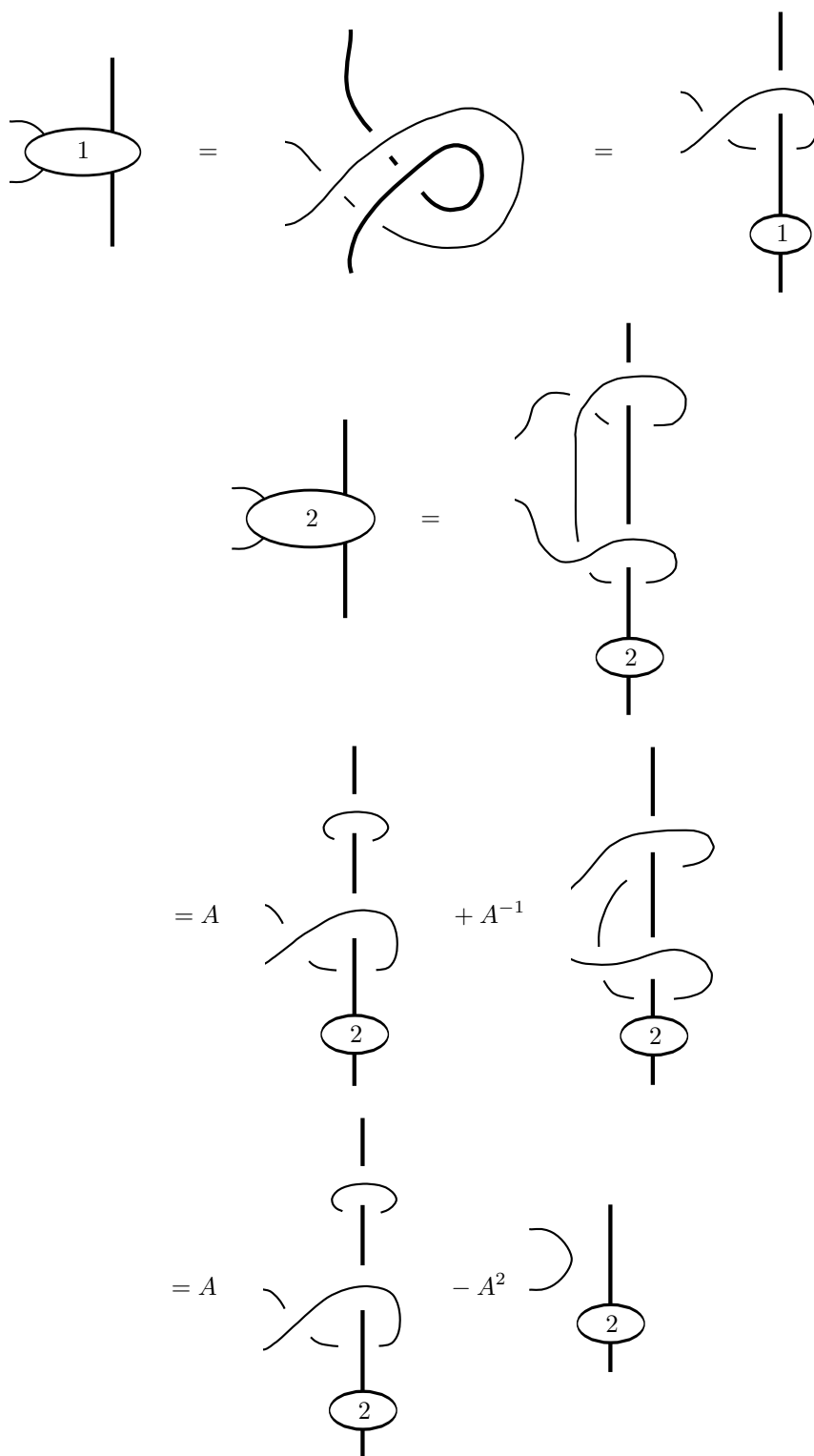


FIGURE 3. Examples of rewriting twists

$$\begin{array}{c} \text{Diagram with loop } n \end{array} = \sum_{j < n} f_j^+(A) \begin{array}{c} \text{Diagram with } j \text{ loops and crossing} \end{array} + \sum_{j < n-1} g_j^+(A) \begin{array}{c} \text{Diagram with } j \text{ loops and loop } n \end{array},$$

where $f_{n-1}^+(A) = A^{n-1}$, and

$$\begin{array}{c} \text{Diagram with loop } -n \end{array} = \sum_{j < n} f_j^-(A) \begin{array}{c} \text{Diagram with } j \text{ loops and crossing} \end{array} + \sum_{j < n-1} g_j^-(A) \begin{array}{c} \text{Diagram with } j \text{ loops and loop } -n \end{array},$$

where $f_{n-1}^-(A) = A^{1-n}$.

Proof. For $n = 1$ and $n = 2$, the result is obtained in Figure 3.

Let $n > 2$, and suppose that the result holds for all $k < n$. Then

$$\begin{aligned}
& \text{Diagram 1} = \text{Diagram 2} = \text{Diagram 3} = \text{Diagram 4} \\
& = A \cdot \text{Diagram 5} + A^{-1} \cdot \text{Diagram 6} \\
& = A \cdot \text{Diagram 7} - A^2 \cdot \text{Diagram 8}
\end{aligned}$$

Hence, the first equation follows by induction on n . The second equation can be obtained by reversing all of the crossings in the first. \square

By rotating the diagrams in the previous lemma by 180 degrees, we also obtain

Lemma 8. *For each integer $n > 0$,*

$$\begin{array}{c} \text{Diagram with a vertical strand passing through an oval labeled } n \text{ with two loops on the right} \end{array} = \sum_{j < n} f_j^+(A) \begin{array}{c} \text{Diagram with a vertical strand passing through an oval labeled } n \text{ with two loops on the left and } j \text{ crossings above} \end{array} + \sum_{j < n-1} g_j^+(A) \begin{array}{c} \text{Diagram with a vertical strand passing through an oval labeled } n \text{ with two loops on the right and } j \text{ crossings above} \end{array},$$

where $f_{n-1}^+(A) = A^{n-1}$, and

$$\begin{array}{c} \text{Diagram with a vertical strand passing through an oval labeled } -n \text{ with two loops on the right} \end{array} = \sum_{j < n} f_j^-(A) \begin{array}{c} \text{Diagram with a vertical strand passing through an oval labeled } -n \text{ with two loops on the left and } j \text{ crossings above} \end{array} + \sum_{j < n-1} g_j^-(A) \begin{array}{c} \text{Diagram with a vertical strand passing through an oval labeled } -n \text{ with two loops on the right and } j \text{ crossings above} \end{array},$$

where $f_{n-1}^-(A) = A^{1-n}$.

In particular, if a component of a link is only twisted about one set of other strands, we obtain, as an immediate corollary of Lemma 7,

Lemma 9. *For each integer $n > 0$,*

$$\begin{array}{c} \text{Diagram with a vertical strand passing through an oval labeled } n \text{ with two loops on the left} \end{array} = \sum_{i \leq n} h_i^+(A) \begin{array}{c} \text{Diagram with a vertical strand passing through an oval labeled } n \text{ with } i \text{ crossings above} \end{array},$$

where $h_n^+(A) = -A^{n+2}$, and

$$\text{Diagram of a loop labeled } -n \text{ on a strand} = \sum_{i \leq n} h_i^-(A) \text{Diagram of } i \text{ loops followed by a loop labeled } -n$$

where $h_n^-(A) = -A^{-n-2}$.

Similarly, as a corollary of Lemma 8,

Lemma 10. For each integer $n > 0$,

$$\text{Diagram of a loop labeled } n \text{ on a strand} = \sum_{i \leq n} h_i^+(A) \text{Diagram of } i \text{ loops followed by a loop labeled } n$$

where $h_n^+(A) = -A^{n+2}$, and

$$\text{Diagram of a loop labeled } -n \text{ on a strand} = \sum_{i \leq n} h_i^-(A) \text{Diagram of } i \text{ loops followed by a loop labeled } -n$$

where $h_n^-(A) = -A^{-n-2}$.

Suppose that a component of a link is twisted with two sets of strands. While more complicated than in the cases previously considered, it is still possible to rewrite the component as a linear combination of loops around the other strands:

Lemma 11. For all integers $m, n > 0$,

$$\begin{aligned}
& \text{Diagram with two vertical lines, each containing an oval labeled } m \text{ and } n \text{ respectively, connected by a horizontal line.} \\
&= \sum_{i \leq m, j \leq n} f_{i,j}^{++}(A) \left(\begin{array}{c} \vdots \\ \text{cup} \\ \vdots \\ \text{cup} \\ \text{oval } m \end{array} \quad \begin{array}{c} \vdots \\ \text{cup} \\ \vdots \\ \text{cup} \\ \text{oval } n \end{array} \right) \\
&+ \sum_{i < m, j < n} g_{i,j}^{++}(A) \left(\begin{array}{c} \vdots \\ \text{cup} \\ \vdots \\ \text{cup} \\ \text{oval } m \end{array} \quad \begin{array}{c} \vdots \\ \text{cup} \\ \vdots \\ \text{cup} \\ \text{oval } n \end{array} \right) ,
\end{aligned}$$

where $f_{m,n}^{++}(A) = -A^{m+n+2}$.

Proof. Applying Lemma 7, and then applying Lemma 8 to each diagram of the resulting linear combination,

$$\begin{aligned}
& \text{Diagram with two vertical lines, each containing an oval labeled } m \text{ and } n \text{ respectively, connected by a horizontal line.} \\
&= \sum_{i < m, j < n} f_i^+(A) f_j^+(A) \left(\begin{array}{c} \vdots \\ \text{cup} \\ \vdots \\ \text{cup} \\ \text{oval } m \end{array} \quad \begin{array}{c} \vdots \\ \text{cup} \\ \vdots \\ \text{cup} \\ \text{oval } n \end{array} \right) \\
&+ \sum_{i \leq m, j < n-1} (-A^3) f_i^+(A) g_j^+(A) \left(\begin{array}{c} \vdots \\ \text{cup} \\ \vdots \\ \text{cup} \\ \text{oval } m \end{array} \quad \begin{array}{c} \vdots \\ \text{cup} \\ \vdots \\ \text{cup} \\ \text{oval } n \end{array} \right)
\end{aligned}$$

$$+ \sum_{i < m, j < n} g_{i,j}^{+-}(A) \begin{array}{c} \text{Diagram with two vertical strands. The left strand has a cap and a circle labeled } m. \text{ The right strand has a cap and a circle labeled } -n. \text{ A horizontal line connects the two strands at the top.} \end{array} \cdot$$

where $g_{m,n}^{+-}(A) = A^{m-n}$.

Proof. Applying Lemma 7, and then applying Lemma 8 to each diagram of the resulting linear combination,

$$\begin{array}{c} \text{Diagram with two vertical strands. The left strand has a circle labeled } m. \text{ The right strand has a circle labeled } -n. \end{array} = \sum_{i < m, j < n} f_i^+(A) f_j^-(A) \begin{array}{c} \text{Diagram with two vertical strands. The left strand has a cap and a circle labeled } m. \text{ The right strand has a cap and a circle labeled } -n. \text{ A wavy line connects the two strands at the top.} \end{array}$$

$$+ \sum_{i \leq m, j < n-1} (-A^3) f_i^+(A) g_j^-(A) \begin{array}{c} \text{Diagram with two vertical strands. The left strand has a cap and a circle labeled } m. \text{ The right strand has a cap and a circle labeled } -n. \end{array}$$

$$+ \sum_{i < m-1, j \leq n} (-A^{-3}) g_i^+ f_j^-(A) \begin{array}{c} \text{Diagram with two vertical strands. The left strand has a cap and a circle labeled } m. \text{ The right strand has a cap and a circle labeled } -n. \end{array}$$

□

Lemma 13. *For all integers $m, n > 0$,*

where $g_{m,n}^{-+}(A) = A^{n-m}$.

Lemma 14. *For all integers $m, n > 0$,*

$$\begin{array}{|c|} \hline \text{Diagram 1: Two vertical lines with two overlapping ovals labeled } -m \text{ and } -n. \\ \hline \end{array}
= \sum_{i \leq m, j \leq n} f_{i,j}^{--}(A)
\begin{array}{|c|} \hline \text{Diagram 2: Two vertical lines. The left line has } i \text{ dots above a cap and an oval labeled } -m \text{ below. The right line has } j \text{ dots above a cap and an oval labeled } -n \text{ below.} \\ \hline \end{array}$$

$$+ \sum_{i < m, j < n} g_{i,j}^{--}(A) \quad \begin{array}{c} \text{Diagram: A vertical line with a horizontal line crossing it from above. Below the horizontal line, there are two vertical lines. The left vertical line has a loop labeled $-m$ at the bottom. The right vertical line has a loop labeled $-n$ at the bottom. The vertical line on the left is labeled i and the vertical line on the right is labeled j . There are vertical dots between the horizontal line and the loops, and between the loops and the bottom of the vertical lines. There are also vertical dots between the two vertical lines at the top and bottom. The entire diagram is enclosed in a large pair of parentheses.$$
 ,

where $f_{i,j}^{--}(A) = -A^{-m-n-2}$.

3. FINITELY GENERATING THE SKEIN MODULE

Since all links in the exterior of the surgery description of $M(\alpha, \beta, \gamma)$ can be isotoped into a genus 2 handlebody and since the skein relations allow us to remove all crossings in a diagram, $S(M(\alpha, \beta, \gamma))$ is generated by $\{x^i y^j z^k\}$.

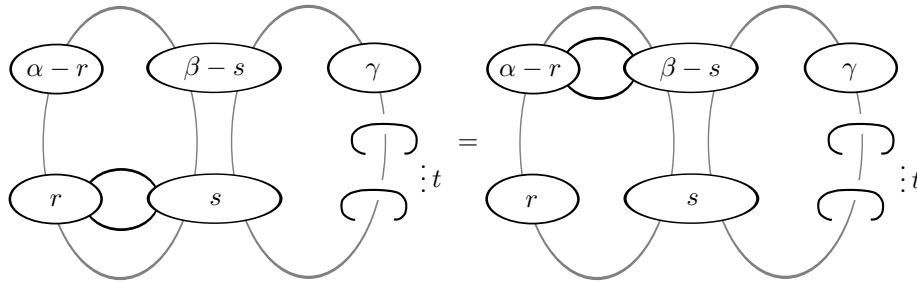
Definition 15. For $a = |\alpha|, b = |\beta|, c = |\gamma| > 0$, we define a strict linear ordering on the generating set $\{x^i y^j z^k\}$ of $M(\alpha, \beta, \gamma)$ as follows: $x^i y^j z^k < x^m y^n z^p$ if

- $\frac{i}{a} + \frac{j}{b} + \frac{k}{c} < \frac{m}{a} + \frac{n}{b} + \frac{p}{c}$,
- $\frac{i}{a} + \frac{j}{b} + \frac{k}{c} = \frac{m}{a} + \frac{n}{b} + \frac{p}{c}$ and $i(k+1) < m(p+1)$,
- $\frac{i}{a} + \frac{j}{b} + \frac{k}{c} = \frac{m}{a} + \frac{n}{b} + \frac{p}{c}$, $i(k+1) = m(p+1)$, and $\max(\frac{j}{b}, \frac{k}{c}) < \max(\frac{n}{b}, \frac{p}{c})$,
- $\frac{i}{a} + \frac{j}{b} + \frac{k}{c} = \frac{m}{a} + \frac{n}{b} + \frac{p}{c}$, $i(k+1) = m(p+1)$, $\max(\frac{j}{b}, \frac{k}{c}) = \max(\frac{n}{b}, \frac{p}{c})$, and $j < n$, or
- $\frac{i}{a} + \frac{j}{b} + \frac{k}{c} = \frac{m}{a} + \frac{n}{b} + \frac{p}{c}$, $i(k+1) = m(p+1)$, $\max(\frac{j}{b}, \frac{k}{c}) = \max(\frac{n}{b}, \frac{p}{c})$, $j = n$, and $k < p$.

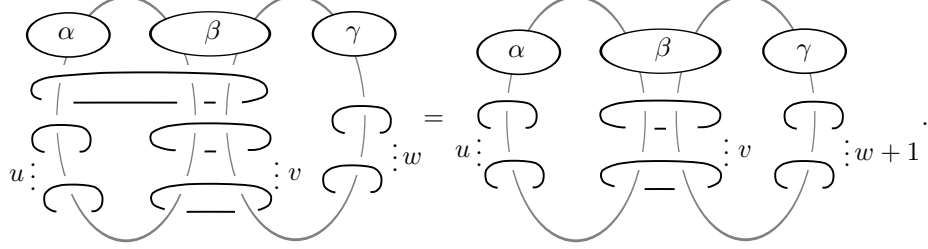
Suppose that $a, b, c > 1$, $\frac{1}{a} < \frac{1}{b} + \frac{1}{c}$, $\frac{1}{b} < \frac{1}{a} + \frac{1}{c}$, and $\frac{1}{c} < \frac{1}{a} + \frac{1}{b}$.

By sliding over an attached 2-handle, we obtain

Definition 16. the Type I relation:



First, note that by Lemmas 11, 12, 13, and 14, each side of the relation can be written as a linear combination of loops of the form $x^i y^j z^k$, since for all nonnegative integers u, v , and w ,



Note that when $r \geq 0$ and $s \geq 0$, the greatest term appearing on the left side of the Type I relation, rewritten as a linear combination of loops, is $x^r y^s z^t$:

When $r, s > 0$, by Lemma 11, $x^r y^s z^t$ and $x^{r-1} y^{s-1} z^{t+1}$ appear as the greatest terms of their respective types.

Since $\frac{1}{c} < \frac{1}{a} + \frac{1}{b}$,

$$\frac{r}{a} + \frac{s}{b} + \frac{t}{c} > \left(\frac{r}{a} + \frac{s}{b} + \frac{t}{c} \right) + \left(-\frac{1}{a} - \frac{1}{b} + \frac{1}{c} \right) = \frac{r-1}{a} + \frac{s-1}{b} + \frac{t+1}{c}.$$

When either $r = 0$ or $s = 0$, the claim follows by Lemma 9 or Lemma 10. When both are 0, the claim follows trivially.

Also note that as long as $r > 0$ or $s > 0$, the leading coefficient is $-A^{r+s+2}$.

Similarly, when $r \leq 0$ and $s \leq 0$, the greatest term appearing on the left side of the Type I relation is $x^{-r} y^{-s} z^t$, and as long as both are not 0, its coefficient is $-A^{r+s-2}$.

When $r > 0$ and $s < 0$, the greatest term appearing on the left side of the Type I relation is $x^{r-1} y^{-s-1} z^{t+1}$:

By Lemma 12, $x^{r-1} y^{-s-1} z^{t+1}$, $x^{r-2} y^{-s} z^t$, and $x^r y^{-s-2} z^t$ appear as the greatest terms of their respective types. Since $\frac{1}{b} < \frac{1}{a} + \frac{1}{c}$,

$$\frac{r-1}{a} + \frac{-s-1}{b} + \frac{t+1}{c} > \left(\frac{r-1}{a} + \frac{-s-1}{b} + \frac{t+1}{c} \right) + \left(-\frac{1}{a} + \frac{1}{b} - \frac{1}{c} \right) = x^{r-2} y^{-s} z^t.$$

Since $\frac{1}{a} < \frac{1}{b} + \frac{1}{c}$,

$$\frac{r-1}{a} + \frac{-s-1}{b} + \frac{t+1}{c} > \left(\frac{r-1}{a} + \frac{-s-1}{b} + \frac{t+1}{c} \right) + \left(\frac{1}{a} - \frac{1}{b} - \frac{1}{c} \right) = x^r y^{-s-2} z^t.$$

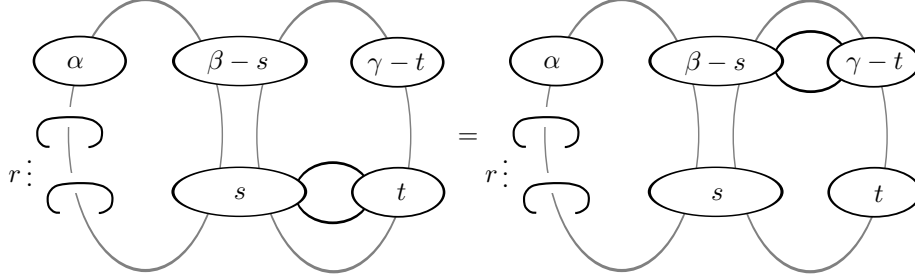
Also note that in this case, the leading coefficient is A^{r+s} .

Similarly, when $r < 0$ and $s > 0$, the greatest term appearing on the left side is $x^{-r-1} y^{s-1} z^{t+1}$, with coefficient A^{r+s} .

Likewise, the greatest term on the right side is $x^{|\alpha-r|-1} y^{|\beta-s|-1} z^{t+1}$, when $\alpha - r$ and $\beta - s$ are nonzero with different signs, and the greatest term on the right side is $x^{|\alpha-r|} y^{|\beta-s|} z^t$ otherwise.

By sliding over the other attached 2-handle, we obtain

Definition 17. *the Type II relation:*



As with the Type I relation, each side of the relation can be rewritten as a linear combination of loops of the form $x^i y^j z^k$.

Also, as with the Type I relation, the greatest term appearing on the left side of the Type II relation is $x^{r+1} y^{|s|-1} z^{|t|-1}$ when the signs of s and t differ, with coefficient A^{s+t} . Otherwise, the greatest term appearing on the left side is $x^r y^{|s|} z^{|t|}$, and as long as one of s and t are nonzero, the leading coefficient is $-A^{s+t \pm 2}$.

Finally, as with the Type I relation, the greatest term on the right side of the Type II relation is $x^{r+1} y^{|\beta-s|-1} z^{|\gamma-t|-1}$ when the signs of $\beta-s$ and $\gamma-t$ differ, and the greatest term appearing on the left side is $x^r y^{|\beta-s|} z^{|\gamma-t|}$ otherwise.

Theorem 18. *For all integers $a, b, c > 1$ such that $\frac{1}{a} < \frac{1}{b} + \frac{1}{c}$, $\frac{1}{b} < \frac{1}{a} + \frac{1}{c}$, and $\frac{1}{c} < \frac{1}{a} + \frac{1}{b}$, $S(M(a, b, c))$ is finitely generated.*

Proof. We show that with respect to our previously defined ordering, $x^i y^j z^k$ can be rewritten as linear combinations of lesser terms whenever $i \geq a$, $j \geq b$, or $k \geq c$. We accomplish this by choosing a Type I or Type II relation in which $x^i y^j z^k$ appears as the greatest term on the left side, as in the previous discussion. We then show that $x^i y^j z^k$ is greater than the greatest term on the right side of the relation. Hence, by subtracting all of the terms less than $x^i y^j z^k$ from both sides of the equation and dividing both sides by the (invertible, as previously discussed) coefficient of $x^i y^j z^k$, we successfully rewrite $x^i y^j z^k$.

Case 1: Suppose $i \geq a$. Let $r = i, s = j, t = k$. Since $r > 0$ and $s \geq 0$, $x^i y^j z^k$ is the greatest term on the left of the Type I relation. Since $a - r = a - i \leq 0$, the greatest term on the right side is $x^{i-a} y^{j-b} z^k$, if $j \geq b$ or $i = a$, and $x^{i-a-1} y^{b-j-1} z^{k+1}$, if $j < b$ and $i > a$.

Case 1.1: Suppose $j \geq b$ or $i = a$. $\frac{i}{a} + \frac{j}{b} + \frac{k}{c} > \frac{i-a}{a} + \frac{j-b}{b} + \frac{k}{c}$, and thus, $x^i y^j z^k > x^{i-a} y^{j-b} z^k$.

Case 1.2: Suppose $j < b$ and $i > a$. $\frac{i}{a} + \frac{j}{b} + \frac{k}{c} > \frac{i}{a} - \frac{j}{b} + \frac{k}{c} = \frac{i-a}{a} + \frac{b-j}{b} + \frac{k}{c} > \left(\frac{i-a}{a} + \frac{b-j}{b} + \frac{k}{c} \right) + \left(-\frac{1}{a} - \frac{1}{b} + \frac{1}{c} \right) = \frac{i-a-1}{a} + \frac{b-j-1}{b} + \frac{k+1}{c}$. Hence, $x^i y^j z^k > x^{i-a-1} y^{b-j-1} z^{k+1}$.

Case 2: Suppose $i < a$ and $j \geq b$. Let $r = i, s = j, t = k$. Since $r \geq 0$ and $s > 0$, $x^i y^j z^k$ is the greatest term on the left of the Type I relation. Since $a - r = a - i > 0$ and $b - s = b - j \leq 0$, the greatest term on the right side is $x^{a-i-1} y^{j-b-1} z^{k+1}$, if $j > b$, and $x^{a-i} z^k$, if $j = b$.

Case 2.1: Suppose $j > b$. $\frac{i}{a} + \frac{j}{b} + \frac{k}{c} > -\frac{i}{a} + \frac{j}{b} + \frac{k}{c} = \frac{a-i}{a} + \frac{j-b}{b} + \frac{k}{c} > \left(\frac{a-i}{a} + \frac{j-b}{b} + \frac{k}{c} \right) + \left(-\frac{1}{a} - \frac{1}{b} + \frac{1}{c} \right) = \frac{a-i-1}{a} + \frac{j-b-1}{b} + \frac{k+1}{c}$, and thus, $x^i y^j z^k > x^{a-i-1} y^{j-b-1} z^{k+1}$.

Case 2.2: Suppose $j = b$. $\frac{i}{a} + \frac{j}{b} + \frac{k}{c} = \frac{i}{a} + 1 + \frac{k}{c} > -\frac{i}{a} + 1 + \frac{k}{c} = \frac{a-i}{a} + \frac{k}{c}$, and hence, $x^i y^j z^k > x^{a-i} z^k$.

Case 3: Suppose $i < a$, $j < b$, and $k \geq c$. Let $r = i, s = j, t = k$. Since $s \geq 0$ and $t > 0$, $x^i y^j z^k$ is the greatest term on the left of the Type II relation. Since $c - t = c - k \leq 0$, the greatest term on the right side is $x^{i+1} y^{b-j-1} z^{k-c-1}$, if $k > c$, and $x^i y^{b-j}$, if $k = c$.

Case 3.1: Suppose $k > c$. Then $\frac{i}{a} + \frac{j}{b} + \frac{k}{c} > \frac{i}{a} - \frac{j}{b} + \frac{k}{c} = \frac{i}{a} + \frac{b-j}{b} + \frac{k-c}{c} > \left(\frac{i}{a} + \frac{b-j}{b} + \frac{k-c}{c}\right) + \left(\frac{1}{a} - \frac{1}{b} - \frac{1}{c}\right) = \frac{i+1}{a} + \frac{b-j-1}{b} + \frac{k-c-1}{c}$, and thus, $x^i y^j z^k > x^{i+1} y^{b-j-1} z^{k-c-1}$.

Case 3.2: Suppose $k = c$. $\frac{i}{a} + \frac{j}{b} + \frac{k}{c} = \frac{i}{a} + \frac{j}{b} + 1 > \frac{i}{a} - \frac{j}{b} + 1 = \frac{i}{a} + \frac{b-j}{b}$, and so, $x^i y^j z^k > x^i y^{b-j}$.

□

Remark 19. Note that we can refine the generating set obtained in the above proof, through additional applications of the Type I and Type II relations. For instance, we can rewrite $x^i y^j z^k$ when

- $i < a$, $j < b$, and $\frac{i}{a} + \frac{j}{b} > 1$,
- $i < a$, $j < b$, $\frac{i}{a} + \frac{j}{b} = 1$, and $i > \frac{a}{2}$,
- $j < b$, $k < c$ and $\frac{j}{b} + \frac{k}{c} > 1$, or
- $j < b$, $k < c$, $\frac{j}{b} + \frac{k}{c} = 1$, and $k > \frac{c}{2}$.

Theorem 20. For all integers $a, b, c > 1$ such that $\frac{1}{a} < \frac{1}{b} + \frac{1}{c}$, $\frac{1}{b} < \frac{1}{a} + \frac{1}{c}$, and $\frac{1}{c} < \frac{1}{a} + \frac{1}{b}$, $S(M(a, -b, c))$ is finitely generated.

Proof. We show that with respect to our previously defined ordering, $x^i y^j z^k$ can be rewritten as linear combinations of lesser terms whenever $i \geq a$, $j \geq b$, or $k > c(2 - \frac{2}{b})$. As in the previous proof, we accomplish this by choosing a Type I or Type II relation in which $x^i y^j z^k$ appears as the greatest term on the left side, and then show that $x^i y^j z^k$ is greater than the greatest term on the right side of the relation. Here, however, the task is a bit more difficult: the difference in signs prevents us from proceeding in a completely straightforward manner.

Case 1: Suppose $i \geq a$. Let $r = i, s = j, t = k$. Since $r > 0$ and $s \geq 0$, $x^i y^j z^k$ is the greatest term on the left of the Type I relation. $a - r = a - i \leq 0$ and $-b - s = -b - j < 0$, and thus $x^{i-a} y^{b+j} z^k$ is the greatest term on the right. $\frac{i}{a} + \frac{j}{b} + \frac{k}{c} = \frac{i-a}{a} + \frac{b+j}{b} + \frac{k}{c}$, and $i(k+1) > (i-a)(k+1)$, so $x^i y^j z^k > x^{i-a} y^{b+j} z^k$.

Case 2: Suppose $i < a$ and $j \geq b$.

Case 2.1: Suppose $k > 0$. Let $r = i+1, s = -j-1, t = k-1$. Since $r > 0$ and $s < 0$, $x^{i+1} y^{-j-1} z^{k-1}$ is the greatest term on the left of the Type I relation. Since $a - r = a - i - 1 \geq 0$ and $-b - s = -b + j + 1 > 0$, $x^{a-i-1} y^{-b+j+1} z^{k-1}$ is the greatest term on the right. $\frac{i}{a} + \frac{j}{b} + \frac{k}{c} > \left(\frac{i}{a} + \frac{j}{b} + \frac{k}{c}\right) + \left(-\frac{1}{a} + \frac{1}{b} - \frac{1}{c}\right) = \frac{a-i-1}{a} + \frac{-b+j+1}{b} + \frac{k-1}{c}$, and thus, $x^i y^j z^k > x^{a-i-1} y^{-b+j+1} z^{k-1}$.

Case 2.2: Suppose $k = 0$.

Case 2.2.1: Suppose $i > 0$. Let $r = i-1, s = -j-1, t = 1$. Since $s < 0$ and $t > 0$, $x^{i-1} y^{-j-1} z$ is the greatest term on the left of the Type II relation. $-b - s = -b + j + 1 > 0$ and $c - t = c - 1 > 0$, and thus $x^{i-1} y^{-b+j+1} z^{c-1}$ is the greatest term on the right. $\frac{i}{a} + \frac{j}{b} + \frac{k}{c} > \left(\frac{i}{a} + \frac{j}{b} + \frac{k}{c}\right) + \left(-\frac{1}{a} + \frac{1}{b} - \frac{1}{c}\right) = \frac{i-1}{a} + \frac{-b+j+1}{b} + \frac{c-1}{c}$, and thus, $x^i y^j > x^{i-1} y^{-b+j+1} z^{c-1}$.

Case 2.2.2: Suppose $i = 0$. Let $r = 0, s = -j, t = 0$. Since $t = 0$, y^j is the greatest term on the left of the Type II relation. $-b - s = -b + j \geq 0$ and $c - t = c > 0$, and thus $y^{-b+j}z^c$ is the greatest term on the right. $\frac{j}{b} = \frac{-b+j}{b} + \frac{c}{c}$ and $0(0+1) = 0(c+1)$. When $j > b$, $\max(\frac{j}{b}, 0) > \max(\frac{-b+j}{b}, \frac{c}{c})$, and when $j = b$, $\max(\frac{j}{b}, 0) = 1 = \max(\frac{-b+j}{b}, \frac{c}{c})$ and $j = b > 0 = -b + j$. Hence, $y^j > y^{-b+j}z^c$.

Case 3: Suppose $i < a, j < b$, and $k > c(2 - \frac{2}{b})$. (Hence, $k > c$.)

Case 3.1: Suppose $i > 0$. Let $r = i - 1, s = -j - 1, t = k + 1$. Since $s < 0$ and $t > 0$, $x^i y^j z^k$ is the greatest term on the left of the Type II relation. Since $-b - s = -b + j + 1 \leq 0$ and $c - t = c - k - 1 < 0$, $x^{i-1} y^{b-j-1} z^{k-c+1}$ is the greatest term on the right. $\frac{i}{a} + \frac{j}{b} + \frac{k}{c} > (\frac{i}{a} + \frac{j}{b} + \frac{k}{c}) + (-\frac{1}{a} - \frac{1}{b} + \frac{1}{c}) = \frac{i-1}{a} + \frac{b-j-1}{b} + \frac{k-c+1}{c}$, and thus, $x^i y^j > x^{i-1} y^{b-j-1} z^{k-c+1}$.

Case 3.2: Suppose $i = 0$.

Case 3.2.1: Suppose $j = b - 1$. Let $r = 1, s = -b, t = k - 1$. Since $r > 0$ and $s < 0$, $y^{b-1} z^k$ is the greatest term on the left of the Type I relation. $a - r = a - 1 > 0$ and $-b - s = 0$, and thus, $x^{a-1} z^{k-1}$ is the greatest term on the right. Since $\frac{b-1}{b} + \frac{k}{c} > (\frac{b-1}{b} + \frac{k}{c}) + (-\frac{1}{a} + \frac{1}{b} - \frac{1}{c}) = \frac{a-1}{a} + \frac{k-1}{c}$, $y^{b-1} z^k > x^{a-1} z^{k-1}$.

Case 3.2.2: Suppose $j < b - 1$. Let $r = 0, s = j, t = k$. Since $s \geq 0$ and $t > 0$, $y^j z^k$ is the greatest term on the left of the Type II relation. $-b - s = -b - j < 0$ and $c - k < 0$, and thus, $y^{b+j} z^{k-c}$ is the greatest term on the right. $\frac{j}{b} + \frac{k}{c} = \frac{b+j}{b} + \frac{k-c}{c}$, $0(k+1) = 0(k-c+1)$, and $\max(\frac{j}{b}, \frac{k}{c}) = \frac{k}{c} > \max(\frac{b+j}{b}, \frac{k-c}{c})$ since $k > c(\frac{2b-2}{b}) \geq c(\frac{b+j}{b})$. Hence $y^j z^k > y^{b+j} z^{k-c}$. \square

Proof of Theorem 4. If α, β , and γ are all positive, the result follows by Theorem 18. If α, β , and γ are all negative, the result follows as well, since $S(M(\alpha, \beta, \gamma))$ is isomorphic to $S(M(-\alpha, -\beta, -\gamma))$.

Suppose that exactly one of α, β , and γ is negative. If $\beta < 0$, the result follows by Theorem 20. If $\alpha < 0$, by sliding the right handle over the left and performing isotopy, we see that $M(\alpha, \beta, \gamma)$ is identical to $M(\gamma, \alpha, \beta)$, and so the result follows. Similarly, if $\gamma < 0$, by sliding the left handle over the right, $M(\alpha, \beta, \gamma)$ is seen to be identical to $M(\beta, \gamma, \alpha)$, and so again the result follows.

If exactly one of α, β , and γ is positive, $S(M(-\alpha, -\beta, -\gamma))$ is finitely generated, and thus $S(M(\alpha, \beta, \gamma))$ is finitely generated as well. \square

4. EXAMPLES

While the previous proofs yield a finite set of generators for $S(M(\alpha, \beta, \gamma))$, they do not exploit the full potential of the Type I and Type II relations. Using the following Python code, we can refine our results for $S(M(a, -b, c))$:

```
def greaterthan(a,b,c,i,j,k,m,n,p):
    if i*b*c + j*a*c + k*a*b > m*b*c + n*a*c + p*a*b:
        return True
    elif i*b*c + j*a*c + k*a*b == m*b*c + n*a*c + p*a*b:
        if i*(k+1) > m*(p+1):
            return True
        elif i*(k+1) == m*(p+1):
            if max(j*c,k*b) > max(n*c,p*b):
```

```

        return True
    elif max(j*c,k*b) == max(n*c,p*b):
        if j > n:
            return True
        elif j == n:
            if k > p:
                return True
    return False

def left1(i,j,k):
    L = []
    if i > 0 or j > 0:
        L.append([i,j,k])
        L.append([-i,-j,k])
    if k > 0:
        L.append([i+1,-j-1,k-1])
        L.append([-i-1,j+1,k-1])
    return L

def left2(i,j,k):
    L = []
    if j > 0 or k > 0:
        L.append([i,j,k])
        L.append([i,-j,-k])
    if i > 0:
        L.append([i-1,j+1,-k-1])
        L.append([i-1,-j-1,k+1])
    return L

def right1(a,b,c,r,s,t):
    if (a-r > 0 and -b-s < 0) or (a-r < 0 and -b-s > 0):
        return [abs(a-r)-1,abs(-b-s)-1,t+1]
    return [abs(a-r),abs(-b-s),t]

def right2(a,b,c,r,s,t):
    if (-b-s > 0 and c-t < 0) or (-b-s < 0 and c-t > 0):
        return [r+1,abs(-b-s)-1,abs(c-t)-1]
    return [r,abs(-b-s),abs(c-t)]

def generatingset(a,b,c):
    GS = []
    MGS = []
    for i in range(a):
        for j in range(b):
            k = 0
            while b*k <= 2*c*(b-1):
                GS.append([i,j,k])
                k += 1

```

```

for T in GS:
    rewrite = False
    for L in left1(T[0], T[1], T[2]):
        R = right1(a, b, c, L[0], L[1], L[2])
        if greaterthan(a, b, c, T[0], T[1], T[2], R[0], R[1], R[2]):
            rewrite = True or rewrite
    for L in left2(T[0], T[1], T[2]):
        R = right2(a, b, c, L[0], L[1], L[2])
        if greaterthan(a, b, c, T[0], T[1], T[2], R[0], R[1], R[2]):
            rewrite = True or rewrite
    if not rewrite:
        MGS.append(T)
return MGS

```

Using the code listed above, we obtain the generating sets listed in the introduction for $S(M(2, -2, 2))$, $S(M(3, -2, 3))$, and $S(M(3, -2, 5))$, and we find that our generating set is minimal for $S(M(2, -2, 2); \mathbb{Z}[A^{\pm 1}], A)$.

As for observing minimality of our generating sets for $S(M(3, -2, 3); R[A^{\pm 1}], A)$ and $S(M(3, -2, 5); R[A^{\pm 1}], A)$, we might hope to consider $S(M(3, -2, 3); R, -1)$ and $S(M(3, -2, 5); R, -1)$, as they are isomorphic to the skein algebras of their fundamental groups, which are generated by representatives of conjugacy classes. For $S(M(3, -2, 3); R[A^{\pm 1}], A)$, however, this will not help, as only three of the conjugacy classes of the binary tetrahedral group are self-inversive: $S(M(3, -2, 3); R, -1)$ can be generated by five elements. See Przytycki and Sikora [11].

Still, for $S(M(3, -2, 5); R[A^{\pm 1}], A)$, we can hope to gain some insight, as its conjugacy classes are self-inversive, and since we have the following result:

Proposition 21. *Suppose that a set $L = \{L_1, \dots, L_n\}$ of links in M represents a generating set for $S(M; R[A^{\pm 1}], A)$.*

- (1) *If L yields a minimal generating set for $S(M; R, -1)$, then L represents a minimal generating set for $S(M; R[A^{\pm 1}], A)$.*
- (2) *If L yields a linearly independent set for $S(M; R, -1)$ and $S(M; R[A^{\pm 1}], A)$ has no $(A + 1)$ torsion, then L represents a basis for $S(M; R[A^{\pm 1}], A)$.*
- (3) *If L yields a linearly independent set for $S(M; R, -1)$ and $S(M; R[A^{\pm 1}], A)$ has torsion, then $S(M; R[A^{\pm 1}], A)$ has $(A + 1)$ torsion.*

Proof. (1) Suppose that $L_n = f_1(A)L_1 + \dots + f_{n-1}(A)L_{n-1}$ in $S(M; R[A^{\pm 1}], A)$. Then in $S(M; R, -1)$, $L_n = f_1(-1)L_1 + \dots + f_{n-1}(-1)L_{n-1}$, a contradiction.

(2) Suppose that $f_1(A)L_1 + \dots + f_n(A)L_n = 0$ in $S(M; R[A^{\pm 1}], A)$. Then in $S(M; R, -1)$, $f_1(-1)L_1 + \dots + f_n(-1)L_n = 0$. L_1, \dots, L_n is a basis of $S(M; R, -1)$, so $f_i(-1) = 0$ for each i , and thus $(A + 1)|f_i$ for each i . Hence, for some g_1, \dots, g_n , $(A + 1)(g_1(A)L_1 + \dots + g_n(A)L_n) = 0$. $S(M; R[A^{\pm 1}], A)$ has no $(A + 1)$ torsion, so $g_1(A)L_1 + \dots + g_n(A)L_n = 0$. Hence, $S(M; R[A^{\pm 1}], A)$ is free.

(3) If L yields a linearly independent set for $S(M; R, -1)$ and $S(M; R[A^{\pm 1}], A)$ has torsion, then L cannot represent a basis, and hence $S(M; R[A^{\pm 1}], A)$ must have $(A + 1)$ torsion by (2). \square

Remark 22. *The existence of torsion is a topic of particular interest in skein theory. For example, McLendon has studied $(A + 1)$ torsion in [9].*

Let G be the binary icosahedral group, with presentation $\langle r, s | r^5 = s^3 = (rs)^2 \rangle$. Since G is finite, the skein algebra of G over \mathbb{C} is isomorphic to $\mathbb{C}[X(G)]$, the $SL(2, \mathbb{C})$ character variety of G ([11], see also Bullock [2]).

Let σ_0 be the trivial 2-dimensional representation of G , let σ_1 be the representation of G that sends r and s to

$$A_1 = \frac{1}{5} \begin{bmatrix} -3e_5 - e_5^2 + e_5^3 - 2e_5^4 & e_5 - 3e_5^2 - 2e_5^3 - e_5^4 \\ e_5 + 2e_5^2 + 3e_5^3 - e_5^4 & -2e_5 + e_5^2 - e_5^3 - 3e_5^4 \end{bmatrix}$$

and

$$B_1 = \frac{1}{5} \begin{bmatrix} -e_5 - 2e_5^2 - 3e_5^3 - 4e_5^4 & 2e_5 - e_5^2 + e_5^3 - 2e_5^4 \\ 2e_5 - e_5^2 + e_5^3 - 2e_5^4 & -4e_5 - 3e_5^2 - 2e_5^3 - e_5^4 \end{bmatrix},$$

respectively, and let σ_2 be the representation of G that sends r and s to

$$A_2 = \begin{bmatrix} e_5 - e_5^2 & -e_5^2 - e_5^4 \\ -e_5 - e_5^4 & -e_5 - e_5^3 \end{bmatrix} \text{ and } B_2 = \begin{bmatrix} 1 & -e_5^3 \\ e_5^2 & 0 \end{bmatrix},$$

respectively, where $e_5 = e^{\frac{2\pi i}{5}}$.

Using GAP [4], we can see that σ_0 , σ_1 , and σ_2 are $SL(2, \mathbb{C})$ representations of G , and any $SL(2, \mathbb{C})$ representation σ of G is equivalent to one of them: if irreducible, σ is equivalent to σ_1 or σ_2 , and if reducible, σ is equivalent to σ_0 , since G is perfect. See Culler and Shalen [3].

Let χ_0 , χ_1 , and χ_2 be the characters of σ_0 , σ_1 , and σ_2 , respectively, and for each $g \in G$, let τ_g be the evaluation map defined on the characters of G by $\tau_g(\chi) = \chi(g)$. Note that since $1, r, r^2, r^3, r^4, r^5, rs, s$, and s^2 represent the conjugacy classes of G , $\mathbb{C}[X(G)]$ is generated by $\tau_1, \tau_r, \tau_{r^2}, \tau_{r^3}, \tau_{r^4}, \tau_{r^5}, \tau_{rs}, \tau_s$, and τ_{s^2} .

| | τ_1 | τ_r | τ_{r^2} | τ_{r^3} | τ_{r^4} | τ_{r^5} | τ_{rs} | τ_s | τ_{s^2} |
|----------|----------|------------------|-----------------|------------------|-----------------|--------------|-------------|----------|--------------|
| χ_0 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| χ_1 | 2 | $-e_5 - e_5^4$ | $e_5^2 + e_5^3$ | $-e_5^2 - e_5^3$ | $e_5 + e_5^4$ | -2 | 0 | 1 | -1 |
| χ_2 | 2 | $-e_5^2 - e_5^3$ | $e_5 + e_5^4$ | $-e_5 - e_5^4$ | $e_5^2 + e_5^3$ | -2 | 0 | 1 | -1 |

From the table, we can see that the following relations hold in $\mathbb{C}[X(G)]$:

- $\tau_{s^2} = 3\tau_s - 2\tau_1$
- $\tau_{rs} = 2\tau_s - \tau_1$
- $\tau_{r^5} = 4\tau_s - 3\tau_1$
- $\tau_{r^4} = 4\tau_s - \tau_r - 2\tau_1$
- $\tau_{r^3} = 3\tau_s - \tau_r - \tau_1$
- $\tau_{r^2} = \tau_s + \tau_r - \tau_1$

Furthermore, $\{\tau_1, \tau_r, \tau_s\}$ are linearly independent in $\mathbb{C}[X(G)]$, since the matrix

$$\begin{bmatrix} \tau_1(\chi_0) & \tau_r(\chi_0) & \tau_s(\chi_0) \\ \tau_1(\chi_1) & \tau_r(\chi_1) & \tau_s(\chi_1) \\ \tau_1(\chi_2) & \tau_r(\chi_2) & \tau_s(\chi_2) \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 2 & -e_5 - e_5^4 & 1 \\ 2 & -e_5^2 - e_5^3 & 1 \end{bmatrix}$$

is invertible.

Thus, $S(M(3, -2, 5); \mathbb{C}, -1)$ is 3-dimensional, and so, we cannot use Proposition 21 to demonstrate that our generating set for $S(M(3, -2, 5); \mathbb{C}[A^{\pm 1}], A)$ is minimal. Hence, we are left with the following:

Question 23. Is $\{1, z, z^2, z^3, z^4, z^5, y, x, x^2\}$ a minimal generating set for $S(M(3, -2, 5); R, A)$ for some ring R and unit A ? If not, is $S(M(3, -2, 5); R, A)$ generated by $\{1, z, x\}$ for every ring R and unit A ?

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